HIGHER-ORDER MULTILINEAR POINCARÉ AND SOBOLEV INEQUALITIES IN CARNOT GROUPS

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ABSTRACT. The notions of higher-order weighted multilinear Poincaré and Sobolev inequalities in Carnot groups are introduced. As an application, weighted Leibniz-type rules in Campanato-Morrey spaces are established.

1. INTRODUCTION AND MAIN RESULTS

Bilinear (and multilinear) Poincaré inequalities such as

$$\begin{aligned}\left(\int_B (|fg - f_B g_B| u)^q\, dx\right)^{1/q} &\leq C \left(\int_B (|\mathbf{Y} f| v_1)^{p_1}\, dx\right)^{1/p_1} \left(\int_B (|g|\, v_2)^{p_2}\, dx\right)^{1/p_2}\\ &+ C \left(\int_B (|f| v_1)^{p_1}\, dx\right)^{1/p_1} \left(\int_B (|\mathbf{Y} g|\, v_2)^{p_2}\, dx\right)^{1/p_2},\end{aligned} \tag{1.1}$$

where $h_B = \frac{1}{|B|}\int_B h(x)\, dx$ and $\mathbf{Y}$ is a collection of vector fields satisfying Hörmander's condition, were introduced and studied in [29] in the context of Carnot-Carathéodory spaces. Such inequalities provide a valid alternative to inequalities of the type

$$\inf_{a\in\mathbb{R}} \left(\int_B |(fg)(x) - a|^q\, dx\right)^{1/q} \leq C \left(\int_B |\nabla(fg)(x)|^p\, dx\right)^{1/p}, \tag{1.2}$$

which fail when $0 < p < 1$. Higher-order versions of (1.2), for instance,

$$\inf_{P(x)} \left(\int_B |(fg)(x) - P(x)|^q\, dx\right)^{1/q} \leq C \left(\int_B |\Delta(fg)(x)|^p\, dx\right)^{1/p}, \tag{1.3}$$

where the supremum is taken over all polynomials $P(x)$ of degree less than two, also fail in general for $0 < p < 1$ (see Remark 3). By Hölder's inequality a natural substitute for (1.3) is given by

$$\begin{aligned}\inf_{P(x)} \left(\int_B |(fg)(x) - P(x)|^q\, dx\right)^{1/q} &\lesssim \left(\int_B |\Delta(f)(x)|^{p_1}\, dx\right)^{1/p_1} \left(\int_B |g(x)|^{p_2}\, dx\right)^{1/p_2}\\ &+ \left(\int_B |\nabla(f)(x)|^{p_1}\, dx\right)^{1/p_1} \left(\int_B |\nabla(g)(x)|^{p_2}\, dx\right)^{1/p_2}\end{aligned} \tag{1.4}$$

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$$+\left(\int_B |f(x)|^{p_1}\,dx\right)^{1/p_1}\left(\int_B |\Delta(g)(x)|^{p_2}\,dx\right)^{1/p_2},$$

where p_1 and p_2 satisfy $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. As we will see, inequality (1.4) and weighted versions of it are indeed true for $1/2<p<1$ (see Theorem 1).

The aim of this work is to introduce higher-order weighted multilinear Poincaré and Sobolev inequalities in the general context of homogeneous Carnot groups that are valid even when $0<p<1$ and are in the spirit of (1.4). The main results in this note are the following (consult Section 2 for definitions).

Theorem 1 (Higher-order weighted multilinear Poincaré inequality)**.** *Suppose $m\in\mathbb{N}$, $\frac{1}{m}<p\le q<\infty$ and $1<p_1,\cdots,p_m<\infty$ such that $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$. Let $\mathbb{G}$ be a homogeneous Carnot group in $\mathbb{R}^n$ of homogeneous dimension Q and n_1 generators, k and m positive integers such that $k\le m\,Q$, d the Carnot-Carathéodory metric in $\mathbb{R}^n$ with respect to a family of generators $\mathbf{X}$ of $\mathbb{G}$. Let u, v_i, $i=1,\cdots,m$, be weights defined on $\mathbb{R}^n$ and satisfying condition* (1.5) *if $q>1$ or condition* (1.6) *if $q\le 1$, where*

(1.5)
$$\sup_{B\ d\text{-ball}} r(B)^{k+Q(1/q-1/p)}\left(\frac{1}{r(B)^Q}\int_B u^{qt}dx\right)^{1/qt}\prod_{i=1}^m\left(\frac{1}{r(B)^Q}\int_B v_i^{-tp_i'}dx\right)^{1/tp_i'}<\infty,$$

for some $t>1$,

(1.6)
$$\sup_{B\ d\text{-ball}} r(B)^{k+Q(1/q-1/p)}\left(\frac{1}{r(B)^Q}\int_B u^{q}dx\right)^{1/q}\prod_{i=1}^m\left(\frac{1}{r(B)^Q}\int_B v_i^{-tp_i'}dx\right)^{1/tp_i'}<\infty,$$

for some $t>1$, with $r(B)$ the radius of B.

Then for all d-ball B and all $\vec{f}=(f_1,\cdots,f_m)\in(C^k(\overline{B}))^m$, there exists a polynomial $P_k(B,\vec{f})$ of degree less than k such that the following weighted m-linear subelliptic Poincaré inequality holds true

(1.7)
$$\left(\int_B\left(\left|\prod_{i=1}^m f_i-P_k(B,\vec{f})\right|u\right)^q dx\right)^{1/q}\le C\sum_{\substack{\alpha_i\in\mathbb{N}_0^{n_1}\\ |\alpha_1|+\cdots+|\alpha_m|=k}}\prod_{i=1}^m\left(\int_B\left(|\mathbf{X}^{\alpha_i}f_i|\,v_i\right)^{p_i}\,dx\right)^{1/p_i},$$

where C is a constant independent of $\vec{f}$ and B.

In the linear case ($m=1$), representation formulas and Poincaré inequalities imply embedding theorems on Campanato-Morrey spaces. These embeddings are applicable when studying the regularity of partial differential equations; see, for instance, Lu [21, 22] where such embeddings were proven in the Carnot-Carathéodory context. The multilinear analogs of these embeddings come in the form of Leibniz-type rules. We next illustrate this by focusing on the bilinear case $m=2$. Indeed, the fractional

Leibniz rule states that for $\alpha > 0$ and $1 < p_1, p_2, q_1, q_2, r < \infty$, with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$, the inequality

$$\| |\nabla|^{\alpha} (fg)\|_{L^r} \lesssim \| |\nabla|^{\alpha} f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \| |\nabla|^{\alpha} g\|_{L^{q_2}} \tag{1.8}$$

holds true, where $\widehat{|\nabla|^{\alpha} h}(\xi) = |\xi|^{\alpha} \hat{h}(\xi)$. Since the pioneering work by Christ-Weinstein [6] and Kenig-Ponce-Vega [18] on the Korteweg-de Vries equation and Kato-Ponce [17] on the Navier-Stokes equation, inequality (1.8) has emerged as an essential tool to study nonlinear PDEs. In particular, PDEs whose nonlinear terms involve quadratic expressions, or, more generally, powers of the solution or its derivatives, and products of the solutions and their derivatives. The use of (1.8) and closely related inequalities has vastly spread across the literature in Analysis and PDEs. Notice that inequality (1.8) is a particular case of

$$\|fg\|_{\mathcal{Z}} \lesssim \|f\|_{\mathcal{X}_1} \|g\|_{\mathcal{Y}_1} + \|f\|_{\mathcal{X}_2} \|g\|_{\mathcal{Y}_2} \tag{1.9}$$

where the spaces $\mathcal{Z} = \dot{W}^{\alpha,r}$, $\mathcal{X}_1 = \dot{W}^{\alpha,p_1}$, $\mathcal{Y}_1 = L^{q_1}$, $\mathcal{X}_2 = L^{p_2}$, and $\mathcal{Y}_2 = \dot{W}^{\alpha,q_2}$ belong to the homogeneous Sobolev scale. As an application of Theorem 1 we derive inequalities of type (1.9) in the scale of weighted Campanato-Morrey spaces in Carnot groups (see Theorem 5). We remark that these estimates are new even in the Euclidean setting.

When $k = 1$ in Theorem 1 one has $P(B, \vec{f}) \equiv \prod_{i=1}^{m} f_{iB}$. Since constants on the right hand side of (1.7) are independent of B , by taking $|B| \to \infty$, we easily obtain the following first-order weighted Sobolev inequality

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} \left(\prod_{i=1}^{m} |f_i|\, u \right)^q dx \right)^{1/q} \\ &\le C \sum_{i=1}^{m} \left(\int_{\mathbb{R}^n} (|\mathbf{X} f_i|\, v_i)^{p_i}\, dx \right)^{1/p_i} \prod_{j \neq i} \left(\int_{\mathbb{R}^n} (|f_j|\, v_j)^{p_j}\, dx \right)^{1/p_j}, \end{aligned}$$

for $f_i \in C_c^k(\mathbb{R}^n)$. More generally, as proved in [27] (see also [23]), the polynomials in Theorem 1 can be taken so that a limit argument gives the following higher-order weighted multilinear Sobolev inequalities.

Theorem 2 (Higher-order weighted multilinear Sobolev inequalities)**.** *Under the same hypothesis as Theorem* 1 *and for* $f_i \in C_c^k(\mathbb{R}^n)$, $i = 1, \cdots, m$,

$$\left(\int_{\mathbb{R}^n} \left(\prod_{i=1}^{m} |f_i|\, u \right)^q dx \right)^{1/q} \le C \sum_{\substack{\alpha_i \in \mathbb{N}_0^{n_1} \\ |\alpha_1| + \cdots + |\alpha_m| = k}} \prod_{i=1}^{m} \left(\int_{\mathbb{R}^n} (|\mathbf{X}^{\alpha_i} f_i|\, v_i)^{p_i}\, dx \right)^{1/p_i}, \tag{1.10}$$

where C is a constant independent of $\vec{f}$.

When $k = 2$, a representation formula in terms of the fundamental solution of a sub-Laplacian on a homogeneous Carnot group and the boundedness properties of the multilinear fractional integrals give the following Sobolev inequality.

Theorem 3 (Second-order weighted multilinear Sobolev inequalities with sub-Laplacians)**.** *Under the same hypothesis as Theorem* 1 *(k=2) and for* $f_i \in C_c^2(\mathbb{R}^n)$*,* $i = 1, \cdots, m$*,*

$$\left(\int_{\mathbb{R}^n} \left(\prod_{i=1}^m |f_i|\, u\right)^q dx\right)^{1/q}$$
$$\leq C \sum_{i=1}^m \left(\int_{\mathbb{R}^n} (|\mathcal{L} f_i|\, v_i)^{p_i}\, dx\right)^{1/p_i} \prod_{j\neq i} \left(\int_{\mathbb{R}^n} (|f_j|\, v_j)^{p_j}\, dx\right)^{1/p_j},$$

where $\mathcal{L}$ *is the sub-Laplacian associated to* $\mathbf{X}$ *and* C *is a constant independent of* $\vec{f}$*.*

Theorem 1, Theorem 2, and Theorem 3 come as an addition to the vast literature on subelliptic Poincaré-type inequalities (including, for instance, [2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 19, 20, 24, 23, 25, 26, 27, 28, 29, 30, 31, 32] and references there in) which address, in particular, the case $p < 1$.

Remark 1. We point out that Theorem 1, as well as the notions of higher-order weighted multilinear Poincaré and Sobolev inequalities, are new even in the Euclidean setting. On the other hand, the Euclidean case of Theorem 2 for $k = 1$ and Theorem 3 were proved in [30].

Remark 2. The multilinear techniques implemented in the proof of Theorem 1 allow for a class of weights strictly larger than the one obtained by iteration of the linear results and Hölder-type inequalities, see [29, p. 10] and [30, Remark 7.5].

Remark 3. We present here an example that shows that inequalities of the type (1.3) fail in general for $0 < p < 1$. The ideas are inspired by the example given in [2] that proves that (1.2) may fail for $0 < p < 1$.

We first consider the one dimensional Euclidean case. Let $\varphi : [-1, 1] \to [0, 1]$ be a continuously differentiable function such that $\varphi(-1) = 0$, $\varphi(1) = 1$, $\varphi'(-1) = \varphi'(1) = 0$ and $\int_{-1}^1 \varphi(x)\, dx = 1$. Set

$$\psi(x) = \begin{cases} 0 & \text{for } x \leq -1, \\ \int_{-1}^x \varphi(t)\, dt & \text{for } -1 \leq x \leq 1, \\ x & \text{for } x \geq 1, \end{cases}$$

For each $\varepsilon \in (0, \frac{1}{2})$, define $f_\varepsilon(x) = \varepsilon\, \psi(\frac{x}{\varepsilon})$ for $x \in \mathbb{R}$. It follows that

$$\lim_{\varepsilon\to 0} \int_{-1}^1 |f_\varepsilon''(x)|^p\, dx = 0, \qquad 0 < p < 1.$$

On the other hand, if $0 < q < \infty$ there exists a constant $C_q > 0$ such that

$$\inf_{a,\, b\in\mathbb{R}} \int_{-1}^1 |f_\varepsilon(x) - (ax + b)|^q\, dx \geq C_q, \qquad \text{for } \varepsilon \in (0, 1/2).$$

Indeed, if $0 < q < \infty$ and $0 < \varepsilon < 1/2$ then

$$\inf_{a,\, b\in\mathbb{R}} \int_{-1}^1 |f_\varepsilon(x) - (ax + b)|^q\, dx \geq C \inf_{a,\, b\in\mathbb{R}} \int_{1/2}^1 (|ax - b| + |(1 - a)x - b|)^q\, dx$$

$$\geq C \inf_{b\in\mathbb{R}} \int_{1/2}^{1} |x-2b|^q \, dx.$$

Elementary computations show that $\int_{1/2}^{1} |x-2b|^q \, dx$ is bounded from below by a positive constant independent of b. For higher dimensions consider $g_\varepsilon(x_1, \cdots, x_n) = f_\varepsilon(x_1)$ where f_ε is as above and integrate, say, in a ball centered at the origin and radius one.

This note consists of two additional sections. We start Section 2 with the necessary background on homogeneous Carnot groups. We then prove our main results, Theorem 1, Theorem 2 and Theorem 3, which follow from two key pieces in the context of homogeneous Carnot groups: the boundedness of multilinear potential operators (Theorem 4) and representation formulas for products of functions (Corollaries 1, 2, 3). In Section 3 we close this article with an application of Theorem 1 to weighted Leibniz-type rules for Campanato-Morrey spaces.

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2. Proofs of Theorem 1, Theorem 2, and Theorem 3

2.1. **Homogeneous Carnot groups.**

2.1.1. *Smooth vector fields.* A smooth vector field X on $\mathbb{R}^n$ is a C^∞ function $X : \mathbb{R}^n \to \mathbb{R}^n$, this is, $X(x) = (a_1(x), \cdots, a_n(x))^T$, $x \in \mathbb{R}^n$, where $a_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \cdots, n$, are infinitely differentiable functions. If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, we denote by Xf the function defined by

$$Xf(x) = X(x)^T \cdot \nabla f(x) = \sum_{j=1}^{n} a_j(x)\, \partial_j f(x), \qquad x \in \mathbb{R}^n.$$

If $\mathbf{X} = \{X_1, \cdots, X_l\}$ is a family of smooth vector fields in $\mathbb{R}^n$, $f \in C^1(\mathbb{R}^n)$, and $\alpha = (\alpha_1, \cdots, \alpha_l) \in \mathbb{N}_0^l$ is a multi-index we define

$$\mathbf{X}^\alpha f := X_1^{\alpha_1}(\cdots X_{l-1}^{\alpha_{l-1}}(X_l^{\alpha_l} f)),$$

and if $k \in \mathbb{N}_0$ we set

$$\left|\mathbf{X}^k f\right| := \left(\sum_{|\alpha|=k} |\mathbf{X}^\alpha f|^2 \right)^{1/2}.$$

2.1.2. *Definition of homogeneous Carnot group.* Let $(\mathbb{R}^n, \diamond)$ be a Lie group on $\mathbb{R}^n$ and denote by $\mathfrak{g}$ its Lie algebra. Consider $n_1, \cdots, n_s \in \mathbb{N}$, $n_1 + \cdots + n_s = n$, and dilations $\{\delta_\lambda\}_{\lambda>0}$ of the form

$$\delta_\lambda(x) = (\lambda\, x^{(1)}, \lambda^2 x^{(2)}, \cdots, \lambda^s x^{(s)}), \quad x^{(i)} \in \mathbb{R}^{n_i}.$$

The triple $\mathbb{G} = (\mathbb{R}^n, \diamond, \delta_\lambda)$ is said to be a homogeneous Carnot group (of step s and n_1 generators) if δ_λ is an automorphism of $(\mathbb{R}^n, \diamond)$ for every $\lambda > 0$ and if the first n_1 elements of the Jacobian basis of $\mathfrak{g}$, say $Z_1, \cdots, Z_{n_1}$, satisfy

$$\text{rank}(\text{Lie}[Z_1, \cdots, Z_{n_1}](x)) = n, \qquad \text{for all } x \in \mathbb{R}^n, \tag{2.1}$$

where $\text{Lie}[Z_1, \cdots, Z_{n_1}]$ is the Lie algebra generated by the vector fields $Z_1, \cdots Z_{n_1}$. The number $Q = \sum_{i=1}^{s} i\, n_i$ is called the homogeneous dimension of $\mathbb{G}$. The vector fields $Z_1, \cdots, Z_{n_1}$ are called the (Jacobian) generators of $\mathbb{G}$, whereas any basis for $\text{span}\{Z_1, \cdots, Z_{n_1}\}$ is called a system of generators of $\mathbb{G}$. It follows that

$$\mathfrak{g} = W^{(1)} \oplus \cdots \oplus W^{(s)},$$

where $W^{(i)}$ denotes the vector space spanned by the commutators of length i of the vectors $Z_1, \cdots, Z_{n_1}$. The elements in $W^{(i)}$ are δ_λ-homogeneous of degree i, and $\dim W^{(i)} = n_i$ if $i \le s$ and $W^{(i)} = \{0\}$ if $i \ge s$. Note that this implies that $\text{Lie}[Z_1, \cdots, Z_{n_1}] = \mathfrak{g}$; moreover condition (2.1) implies that $\{Z_1, \cdots Z_{n_1}\}$ is a family of vector fields satisfying Hörmander's condition.

The second order differential operator

$$\mathcal{L} = \sum_{j=1}^{n_1} X_i^2$$

is called the canonical sub-Laplacian of $\mathbb{G}$ if $X_i = Z_i$, $i = 1, \cdots, n_1$, and simply a sub-Laplacian if $\{X_1, \cdots, X_{n_1}\}$ is a system of generators of $\mathbb{G}$. We point out that there are characterizations of families of smooth vector fields $\{X_1, \cdots, X_{n_1}\}$ for which there exists a homogeneous Carnot group with respect to which $\sum_{i=1}^{n_1} X_i^2$ is a sub-Laplacian (see, for instance, [1, p.191]).

If $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$ is a multi-index and $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ we set

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n, \quad |\alpha|_{\mathbb{G}} = \sigma_1\alpha_1 + \sigma_2\alpha_2 + \cdots + \sigma_n\alpha_n,$$

where $\sigma_i = 1$ for $i = 1, \cdots, n_1$, $\sigma_i = 2$ for $i = n_1 + 1, \cdots, n_1 + n_2$, $\sigma_i = 3$ for $i = n_1+n_2+1, \cdots, n_1+n_2+n_3$, and so on. If $P(x) = \sum_\alpha c_\alpha x^\alpha$ is a polynomial on $\mathbb{G}$ the homogeneous degree (or just degree) of P is defined as $\deg_{\mathbb{G}}(P) = \max\{|\alpha|_{\mathbb{G}} : c_\alpha \neq 0\}$.

2.2. **Multilinear fractional integrals in Carnot groups.** Given a Carnot group $\mathbb{G} = (\mathbb{R}^n, \diamond, \delta_\lambda)$ of homogeneous dimension Q and a system of generators of $\mathbb{G}$, say $\mathbf{X} = \{X_1, \cdots, X_{n_1}\}$, we consider on $\mathbb{R}^n$ the Carnot-Carathéodory metric d associated to $\mathbf{X}$. If $B_d(x, r)$ is the d-ball of radius r centered at x then $|B_d(x, r)| = c_d\, r^Q$ where $c_d = |B_d(0, 1)|$ (see [1, p. 248]). It then follows that $(\mathbb{R}^n, d, \text{Lebesgue measure})$ is a space of homogeneous type. For $\vec{x} = (x_1, \cdots, x_m)$ and $\vec{y} = (y_1, \cdots, y_m)$ with $x_i, y_i \in \mathbb{R}^n$, $i = 1, \cdots, m$, we define $d(\vec{x}, \vec{y}) := d(x_1, y_1) + \cdots + d(x_m, y_m)$; in the case when $x_1 = x_2 = \cdots = x_m =: x$, we simply write $d(x, \vec{y})$ instead of $d(\vec{x}, \vec{y}.)$

In the framework of Carnot groups, we define the multilinear fractional integral of order $\tau > 0$ by

$$\mathcal{I}_{\mathbb{G},\tau}(\vec{f})(x) := \int_{\mathbb{R}^{nm}} \frac{\vec{f}(\vec{y})}{d(x, \vec{y})^{m\,Q-\tau}}\, d\vec{y}, \qquad x \in \mathbb{R}^n, \tag{2.2}$$

where $\vec{f} = (f_1, \cdots, f_m)$, $f_i : \mathbb{R}^n \to \mathbb{R}$, and for $\vec{y} = (y_1, \cdots, y_m) \in \mathbb{R}^{nm}$, $\vec{f}(\vec{y}) = \prod_{i=1}^m f_i(y_i)$.

Theorem 4. *Suppose $m \in \mathbb{N}$, $\frac{1}{m} < p \le q < \infty$ and $1 < p_1, \cdots, p_m < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Let $\mathbb{G}$ be a homogeneous Carnot group in $\mathbb{R}^n$ of homogeneous dimension Q, τ a positive real number, m a positive integer such that $\tau \le m\,Q$, d the Carnot-Carathéodory metric in $\mathbb{R}^n$ with respect to a family of generators $\mathbf{X}$ of $\mathbb{G}$, and u, v_i, $i = 1, \cdots, m$, weights defined on $\mathbb{R}^n$ and satisfying condition* (1.5) *if $q > 1$ or condition* (1.6) *if $q \le 1$ with k replaced by τ. Then there exists a constant C such that*

$$\left(\int_{\mathbb{R}^n} \left(\left|\mathcal{I}_{\mathbb{G},\tau}(\vec{f})(x)\right| u(x)\right)^q dx\right)^{1/q} \le C \prod_{i=1}^m \left(\int_{\mathbb{R}^n} (|f_i(x)|\, v_i(x))^{p_i} dx\right)^{1/p_i}$$

for all $\vec{f} = (f_1, \cdots, f_m) \in L^{p_1}(\mathbb{R}^n, v_1^{p_1} dx) \times \cdots \times L^{p_m}(\mathbb{R}^n, v_m^{p_m} dx)$. The constant C depends only on structural constants and the constants appearing in (1.5) *and* (1.6).

Proof. Since $(\mathbb{G}, d, \text{Lebesgue measure})$ is a space of homogeneous type we just need to check that the hypothesis of [29, Corollary 1] are satisfied if $\tau \le m\,Q$. The conditions to be checked are the reverse doubling property of Lebesgue measure with respect to d-balls and a growth condition for the kernel of the multilinear fractional integral.

The reverse doubling condition in this setting means that there are positive constants c and δ such that

$$\frac{|B_d(x_1, r_1)|}{|B_d(x_2, r_2)|} \ge c \left(\frac{r_1}{r_2}\right)^{\delta},$$

whenever $B_d(x_2, r_2) \subset B_d(x_1, r_1)$, $x_1, x_2 \in \mathbb{R}^n$, and $0 < r_1,\ r_2 < \infty$. Since $|B_d(x, r)| = c_d\, r^Q$ the above inequality holds true with a uniform constant c and any positive $\delta \le Q$.

The growth condition for the kernel in this context means that for every positive constant C_1 there exists a positive constant C_2 such that for all $\vec{x}$, $\vec{y}$, $\vec{z} \in \mathbb{R}^{nm}$,

$$\begin{aligned} \frac{d(\vec{x}, \vec{y})^{\tau}}{\prod_{i=1}^m |B_d(x_i, d(\vec{x}, \vec{y}))|} &\le C_2 \frac{d(\vec{z}, \vec{y})^{\tau}}{\prod_{i=1}^m |B_d(z_i, d(\vec{z}, \vec{y}))|}, \quad d(\vec{z}, \vec{y}) \le C_1\, d(\vec{x}, \vec{y}) \\ \frac{d(\vec{x}, \vec{y})^{\tau}}{\prod_{i=1}^m |B_d(x_i, d(\vec{x}, \vec{y}))|} &\le C_2 \frac{d(\vec{y}, \vec{z})^{\tau}}{\prod_{i=1}^m |B_d(y_i, d(\vec{y}, \vec{z}))|}, \quad d(\vec{y}, \vec{z}) \le C_1\, d(\vec{x}, \vec{y}). \end{aligned}$$

Both inequalities follow from the facts that $|B_d(x, r)| = c_d\, r^Q$ and $\tau \le m\,Q$. □

2.3. **Multilinear higher-order representation formulas in Carnot groups.** In this subsection we prove multilinear higher-order representation formulas for a family of generators of a homogeneous Carnot group (Corollaries 1, 2, 3) as consequences of their linear counterparts (Theorems **A**, **B**, **C**).

Theorem A ([26, p.111, Corollary E])**.** *Let $\mathbb{G}$ be a homogeneous Carnot group in $\mathbb{R}^n$ of homogeneous dimension Q, k a positive integer, d the Carnot-Carathéodory metric*

in $\mathbb{R}^n$ with respect to a family of generators $\mathbf{X}$ of $\mathbb{G}$, B a d-ball, and $f \in C^k(B)$. Then there exists a polynomial $P_k(B,f)$ of degree less than k such that for $x \in B$,

$$|f(x) - P_k(B,f)(x)| \le C \int_B \left|\mathbf{X}^k f(y)\right| \frac{d(x,y)^k}{|B_d(x,d(x,y))|}\,dy + C\,\frac{r(B)^k}{|B|}\int_B \left|\mathbf{X}^k f(y)\right|\,dy,$$

where C is independent of $\vec{f}$, x and B. Moreover, if $k \le Q$ then

$$|f(x) - P_k(B,f)(x)| \le C \int_B \left|\mathbf{X}^k f(y)\right| \frac{d(x,y)^k}{|B_d(x,d(x,y))|}\,dy. \tag{2.3}$$

Corollary 1 (Higher-order multilinear representation formula.). *Let $\mathbb{G}$ be a homogeneous Carnot group in $\mathbb{R}^n$ of homogeneous dimension Q and n_1 generators, k and m positive integers such that $k \le m\,Q$, d the Carnot-Carathéodory metric in $\mathbb{R}^n$ with respect to a family of generators $\mathbf{X}$ of $\mathbb{G}$, B a d-ball, and $\vec{f} = (f_1, \cdots, f_m) \in (C^k(B))^m$. Then there exists a polynomial $P_k(B,\vec{f})$ of degree less than k such that for $x \in B$,*

$$\left|\prod_{i=1}^m f_i(x) - P_k(B,\vec{f})(x)\right| \le C \sum_{\substack{\alpha_i \in \mathbb{N}_0^{n_1} \\ |\alpha_1|+\cdots+|\alpha_m|=k}} \mathcal{I}_{\mathbb{G},k}(|\mathbf{X}^{\alpha_1} f_1|\,\chi_B, \cdots, |\mathbf{X}^{\alpha_m} f_m|\,\chi_B)(x), \tag{2.4}$$

where C is independent of $\vec{f}$, x and B.

Proof. Consider the Carnot group $\mathbb{G}^{(m)}$ in $\mathbb{R}^{nm}$ given by the sum of m copies of $\mathbb{G}$ (see [1, p. 190]) and note that $\mathbb{G}^{(m)}$ has homogeneous dimension $m\,Q$. Let $\tilde{\mathbf{X}}$ be the family of generators for $\mathbb{G}^{(m)}$ given by m copies of $\mathbf{X}$ with appropriately added zeros and $\tilde{d}$ be the Carnot-Carathéodory metric in $\mathbb{R}^{nm}$ associated with $\tilde{\mathbf{X}}$. Then $B_{\tilde{d}}(\vec{x},r) = \prod_{i=1}^m B_d(x_i,r)$ for $r > 0$ and $\vec{x} = (x_1, \cdots, x_m) \in \mathbb{G}^{(m)}$ (see [24, Lemma 1]). Therefore B^m is a $\tilde{d}$-ball and Theorem **A** applied to $f(\vec{y}) = \prod_{i=1}^m f_i(y_i)$, $\vec{y} = (y_1, \cdots, y_m)$, gives that there exists a polynomial $P_k(B^m, f)(\vec{x})$ on $\mathbb{G}^{(m)}$ of degree less than k such that for all $\vec{x} = (x_1, \cdots, x_m) \in B^m$

$$\begin{aligned}\left|\prod_{i=1}^m f_i(x_i) - P_k(B^m,f)(\vec{x})\right| &\le C \int_{B^m} \left|\tilde{\mathbf{X}}^k f(\vec{y})\right| \frac{\tilde{d}(\vec{x},\vec{y})^k}{\left|B_{\tilde{d}}(\vec{x},\tilde{d}(\vec{x},\vec{y}))\right|}\,d\vec{y}, \\ &= C \int_{B^m} \left|\tilde{\mathbf{X}}^k f(\vec{y})\right| \frac{\tilde{d}(\vec{x},\vec{y})^k}{\prod_{i=1}^m \left|B_d(x_i,\tilde{d}(\vec{x},\vec{y}))\right|}\,d\vec{y},\end{aligned}$$

where C is independent of f, $\vec{x}$ and B. Restricting to the diagonal, $x_1 = x_2 = \cdots = x_m = x \in B$, setting $P_k(B,\vec{f})(x) := P_k(B^m,f)(\vec{x})$, which is a polynomial in x of degree less than k, and using that $\tilde{d}(\vec{x},\vec{y}) \sim d(x,\vec{y})$ and $|B_d(x,r)| = c_d\,r^Q$ we obtain

$$\left|\prod_{i=1}^m f_i(x) - P_k(B,\vec{f})(x)\right| \le C \int_{B^m} \frac{|\tilde{\mathbf{X}}^k f(\vec{y})|}{d(x,\vec{y})^{mQ-k}}\,d\vec{y}$$

$$\leq C \sum_{\substack{\alpha_i \in \mathbb{N}_0^{n_1} \\ |\alpha_1|+\cdots+|\alpha_m|=k}} \int_{B^m} \frac{|\mathbf{X}^{\alpha_1} f_1(y_1) \cdots \mathbf{X}^{\alpha_m} f_m(y_m)|}{d(x,\vec{y})^{mQ-k}} d\vec{y}$$

showing (2.5). □

In [27] it is proved that the polynomial in (2.3) can be taken in such a way that $|B| \to \infty$ gives the following global representation formula.

Theorem B ([27, p. 659, Theorem 3.1])**.** *Let $\mathbb{G}$ be a homogeneous Carnot group in $\mathbb{R}^n$ of homogeneous dimension Q, k a positive integer such that $k \leq Q$, and d the Carnot-Carathéodory metric in $\mathbb{R}^n$ with respect to a family of generators $\mathbf{X}$ of $\mathbb{G}$. Then there exists a constant C such that*

$$|f(x)| \leq C \int_{\mathbb{R}^n} \left|\mathbf{X}^k f(y)\right| \frac{d(x,y)^k}{|B_d(x,d(x,y))|}\, dy, \qquad f \in C_c^k(\mathbb{R}^n),\ x \in \mathbb{R}^n.$$

Reasoning as in the proof of Corollary 1 we obtain

Corollary 2 (Higher-order global multilinear representation formula.)**.** *Let $\mathbb{G}$ be a homogeneous Carnot group in $\mathbb{R}^n$ of homogeneous dimension Q and n_1 generators, k and m positive integers such that $k \leq m\,Q$, and d the Carnot-Carathéodory metric in $\mathbb{R}^n$ with respect to a family of generators $\mathbf{X}$ of $\mathbb{G}$. There exists a constant C such that*

(2.5)
$$\left|\prod_{i=1}^m f_i(x)\right| \leq C \sum_{\substack{\alpha_i \in \mathbb{N}_0^{n_1} \\ |\alpha_1|+\cdots+|\alpha_m|=k}} \mathcal{I}_{\mathbb{G},k}(|\mathbf{X}^{\alpha_1} f_1|, \cdots, |\mathbf{X}^{\alpha_m} f_m|)(x), \quad f_i \in C_c^k(\mathbb{R}^n),\ x \in \mathbb{R}^n.$$

The following representation formula is well known (see, for instance, [1, p.236]).

Theorem C. *Let $\mathcal{L}$ be a sub-Laplacian on a homogeneous Carnot group $\mathbb{G}$ in $\mathbb{R}^n$ of homogeneous dimension strictly larger than 2. If Γ is the fundamental solution for $\mathcal{L}$ then*

$$\phi(x) = -\int_{\mathbb{R}^n} \Gamma(x^{-1} \diamond y)\, \mathcal{L}\phi(y)\, dy, \qquad \phi \in \mathcal{C}_0^\infty(\mathbb{R}^n),\ x \in \mathbb{R}^n.$$

Remark 4. The homogeneous dimension of $\mathbb{G}$ being strictly larger than 2 guaranties the existence of a fundamental solution for $\mathcal{L}$, which is unique. Also $\Gamma(x) \sim d(x,0)^{2-Q}$, $x \neq 0$, where d is the Carnot-Carathéodory metric in $\mathbb{R}^n$ associated with the family of generators corresponding to the sub-Laplacian $\mathcal{L}$.

Corollary 3 (Second-order global multilinear representation formula with a sub--Laplacian.)**.** *Let $\mathbb{G}$ be a homogeneous Carnot group in $\mathbb{R}^n$ of homogeneous dimension Q, m a positive integer such that $m\,Q > 2$, $\mathbf{X}$ a family of generators of $\mathbb{G}$ and $\mathcal{L}$ its sub-Laplacian. There exists a constant C such that*

$$\left|\prod_{i=1}^m f_i(x)\right| \leq C \sum_{i=1}^m \mathcal{I}_{\mathbb{G},2}(|f_1|, \cdots, |\mathcal{L} f_i|, \cdots, |f_m|)(x), \tag{2.6}$$

for all $x \in \mathbb{R}^n$ *and* $f_i \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $i = 1, \cdots, m$.

Proof. Let d be the Carnot-Carathéodory metric in $\mathbb{R}^n$ with respect to $\mathbf{X}$ and let $\mathbb{G}^{(m)}$, $\tilde{\mathbf{X}}$ and $\tilde{d}$ be as in the proof of Corollary 1. If $\tilde{\Gamma}$ is the fundamental solution for the sub-Laplacian $\tilde{\mathcal{L}}$ corresponding to $\tilde{\mathbf{X}}$ then Theorem **B** gives

$$f_1(x_1)f_2(x_2)\cdots f_m(x_m) = -\int_{\mathbb{R}^{mn}} \tilde{\Gamma}((\vec{x})^{-1}\tilde{\diamond}\vec{y})\,\tilde{\mathcal{L}}(f_1\cdots f_m)(\vec{y})\,d\vec{y},$$

for $\vec{x} = (x_1, \cdots, x_m) \in \mathbb{R}^{mn}$, $\tilde{\diamond}$ the group operation in $\mathbb{G}^{(m)}$ and $f_i \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $i = 1, \cdots, m$.

We note that $\tilde{\Gamma}((\vec{x})^{-1}\diamond\vec{y}) \sim \tilde{d}(\vec{x},\vec{y})^{2-mQ} \sim d(\vec{x},\vec{y})^{2-mQ}$ and that $\tilde{\mathcal{L}}(f_1\cdots f_m)(\vec{y}) = \sum_{i=1}^m \mathcal{L}f_i(y_i)\cdot\Pi_{j\neq i}f_j(y_j)$ for $\vec{y} = (y_1, \cdots, y_m) \in \mathbb{R}^{mn}$. By these remarks and by taking $x := x_1 = x_2 = \cdots = x_m$ in the above formula we obtain (2.6). □

Finally, the proofs of Theorems 1, 2, and 3 follow from Theorem 4 and Corollaries 1, 2 and 3, respectively.

3. An application to weighted Leibniz-type rules in Campanato-Morrey spaces

Let $\mathbb{G}$ be a homogeneous Carnot group in $\mathbb{R}^n$, d the Carnot-Carathéodory metric in $\mathbb{R}^n$ with respect to a family of generators $\mathbf{X}$ of $\mathbb{G}$, $w \geq 0$ a weight and $p, \lambda > 0$. A function $f \in L^1_{loc}(\mathbb{R}^n, w^p)$ is said to belong to the weighted Morrey space $L^{p,\lambda}(w)$ if

$$\|f\|_{L^{p,\lambda}(w)} = \sup_B \left(\frac{1}{|B|^{\lambda/n}}\int_B |f(x)w(x)|^p\,dx\right)^{1/p} < \infty$$

where the supremum is over all d-balls B. Next, we define the weighted Campanato space of order k, $\mathcal{L}_k^{p,\lambda}(w)$. Let $\mathcal{P}_k$ be the collection of polynomials in $\mathbb{G}$ of degree less than k. We write $f \in \mathcal{L}_k^{p,\lambda}(w)$ if

$$\|f\|_{\mathcal{L}_k^{p,\lambda}(w)} = \sup_B \inf_{P\in\mathcal{P}_k} \left(\frac{1}{|B|^{\lambda/n}}\int_B (|f(x)-P(x)|w(x))^p\,dx\right)^{1/p} < \infty.$$

As a consequence of Theorem 1 we have the following.

Theorem 5. *Let* $1 < p_1, p_2 < \infty$, p *be defined by* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $q \geq p$ *and* $\lambda, \lambda_1, \lambda_2 \in (0,\infty)$ *be such that* $\frac{\lambda}{q} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}$. *If* $\mathbb{G}$ *is a homogeneous Carnot group in* $\mathbb{R}^n$ *of homogeneous dimension* Q *and* n_1 *generators,* k *is a positive integer such that* $k \leq 2\,Q$, d *is the Carnot-Carathéodory metric in* $\mathbb{R}^n$ *with respect to a family of generators* $\mathbf{X}$ *of* $\mathbb{G}$ *and* u, v_1, v_2 *are weights on* $\mathbb{R}^n$ *satisfying condition* (1.5) *if* $q > 1$ *or condition* (1.6) *if* $q \leq 1$ *with* $m = 2$, *then*

$$\|fg\|_{\mathcal{L}_k^{q,\lambda}(u)} \leq C \sum_{\substack{\alpha_i\in\mathbb{N}_0^{n_1}\\ |\alpha_1|+|\alpha_2|=k}} \|\mathbf{X}^{\alpha_1}f\|_{L^{p_1,\lambda_1}(v_1)}\,\|\mathbf{X}^{\alpha_2}g\|_{L^{p_2,\lambda_2}(v_2)}.$$

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